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Gauge differential form theories on the lattice

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Abstract. We discuss gauge differential form theories on the lattice which have Kalb–Ramond theories of interacting r and $r-1$ forms as naïve continuum limit; the Higgs character of the $r-1$ form is clearly displayed. We derive some inequalities useful for drawing tentative phase diagrams and analyse the existence of θ-vacua.

1. Introduction

Multi-index fields were introduced by Kalb and Ramond (1974) as a generalisation of the electromagnetic interaction between point-like charges to the case of extended objects. Recently, multi-index fields have received particular attention in a few sectors of high energy physics, especially in the U(1) problem in QCD and in supergravity (Townsend 1981).

The purpose of this paper is to discuss the lattice regularisation of these fields with r-indices where an interaction with $r-1$ fields is also introduced. Some inequalities are obtained for the critical temperatures at different dimensions and number of indices which may be used for discussing phase diagrams. Furthermore, we generalise θ-vacua of the scalar quantum electrodynamics in two dimensions to the case of r-index fields in $r+1$ dimensions.

What appears at this stage is that the conclusions we draw depend qualitatively only on $d-r$ when $r \geqslant 1$. This is in contradiction with some approximations, e.g. the Migdal one (Orland 1982), and seems also to be in contrast with heuristic arguments based on Hausdorff dimension (for similar arguments see Parisi (1979)).

Throughout the discussion we avoid the cumbersome notation where all the indices appear, using the formalism of differential forms (see e.g. Guth 1980).

The paper has the following subdivision. In § 2 we review briefly the classical theory on the continuum and on the lattice. In § 3 we derive some correlation inequalities for the Wilson–Wegner loop and for the critical coupling at different d (dimension) and r (rank). A diamagnetic inequality is also obtained. In § 4 the generalisation of θ-vacua is discussed. We end with some conclusions in § 5.

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2. Antisymmetric gauge field on the lattice

The classical theory of the r-index fields is described by an action:

$$S = \tfrac{1}{2} \int F \wedge {}^*F \equiv \int F_{\mu_1 \ldots \mu_{r+1}} F^{\mu_1 \ldots \mu_r+1} \frac{d^d x}{2r!} \tag{1}$$

where the field strength F is an $(r+1)$-form defined starting from a potential A in analogy with electrodynamics ($r=1$):

$$F = dA \tag{2}$$

with

$$A = A_{\mu_1 \ldots \mu_2} \, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r}. \tag{3}$$

Due to equation (2) we have that F is invariant under local gauge transformation

$$A \to A' = A + d\theta, \tag{4}$$

θ being an $(r-1)$-form.

We shall call this class of fields gauge differential forms of rank r. When $r=1$ is taken, we get the well known electrodynamics, while for $r=2$ we have the Kalb and Ramond fields (Kalb and Ramond 1974). More precisely, they introduce also an interaction with a 'Higgs boson' B_μ in the following way:

$$S = \int d^d x \, (\partial_{[\mu} A_{\nu\rho]} \partial^{[\mu} A^{\nu\rho]} + g^2 A_{\mu\nu} A^{\mu\nu} + 2g A^{\mu\nu} \partial_{[\mu} B_{\nu]} + \partial_{[\mu} B_{\nu]} \partial^{[\mu} B^{\nu]}) \tag{5}$$

which is invariant under the gauge transformation

$$A_{\mu\nu} \to A'_{\mu\nu} = A_{\mu\nu} + \partial_{[\mu}\theta_{\nu]}, \qquad B_\mu \to B'_\mu = B_\mu - g\theta_\mu. \tag{6}$$

We note that the transformation law for B_μ is more similar to a Higgs field than to a gauge field (hence its name).

It is straightforward to generalise the action (2.5) to gauge fields A of rank r and Higgs fields B of rank $r-1$. In compact form we have for the action

$$S = \int dA \wedge {}^*dA + g^2 A \wedge {}^*A + 2gA \wedge {}^* dB + dB \wedge {}^* dB \tag{7}$$

which is invariant under the gauge transformation

$$A \to A' = A + d\theta, \qquad B \to B' = B - g\theta, \tag{8}$$

where θ is an arbitrary $(r-1)$ form.

It is well known that compactification of the symmetry group in electrodynamics gives rise to problems due to the introduction of monopole-type singularities. When antisymmetric tensors are considered, problems are already present at an early stage. In fact, while for the case $r=1$ we have to deal with the gauge function (see the θ function which appears in the gauge transformation (4)) and it is clear what we mean by U(1)-valued gauge functions, now, when $r>1$, it is not obvious what U(1)-valued $(r-1)$-forms mean. However, this difficulty may be overcome by reinterpreting the theory in an $(r-1)$-times iterated loop space, (Freund and Nepomechie 1981]. In such a space, the topological properties of the A and B fields and the Higgs character of the latter are clearly put in evidence (Marchetti 1982). Now we would like to discuss such a theory in its lattice regularisation. It is known that, in such a case, singularity problems no longer exist, and the same is true for the problem of what 'U(1)-valued $r-1$ differential forms' means, due to the definition of the fields. The solution of the latter problem is related to the growth of the cell dimension, where the field is defined, as the rank of the form increases.

We recall now the basic definition of the formalism of differential forms on the lattice. Let Λ be a d-dimensional hypercubic lattice and let C_r denote an elementary oriented cell whose dimension is r. The cell with reversed orientation will be denoted by $-C_r$. An r-form A is a map from the set of r-cells into $\mathbb{R}$ (in general this can be a ring, e.g. $\mathbb{Z}$, U(1) etc),

$$A:\quad C_r \to A(C_r), \tag{9}$$

with the property

$$A(C_r) = -A(-C_r). \tag{10}$$

Furthermore, one defines the (co)boundary operator, $(\delta)d$ which maps r-forms into $(r-1)\; r+1$ forms in the following way:

$$\mathrm{d}A(C_{r+1}) = \sum_{C_r \subset \partial C_{r+1}} A(C_r), \tag{11}$$

$$\delta A(C_{r-1}) = \sum_{C_r : C_{r-1} \subset \partial C_r} A(C_r). \tag{12}$$

Moreover, one defines the inner product of r-forms:

$$(A', A) = \sum_{C_r} \overline{A'(C_r)} A(C_r). \tag{13}$$

With this definition, the operators δ and d are adjoints with respect to the product defined by equation (13). The action in the Wilson form for the interacting system is

$$S = \beta_r \sum_{C_{r+1}} \cos \mathrm{d}A(C_{r+1}) + \lambda_{r-1} \sum_{C_r} \cos[\mathrm{d}B(C_r) + qA(C_r)] \tag{14}$$

where $A, B \in [-\pi, \pi]$ and q is an integer which is the charge of the B fields. Equation (14) can be rewritten in an equivalent form, defining the following fields taking values onto the $U(1)$ group:

$$U(C_r) = \exp[\mathrm{i}A(C_r)], \qquad \phi(C_{r-1}) = \exp[\mathrm{i}B(C_{r-1})q]. \tag{15a, b}$$

Thus, equation (14) becomes

$$S = \tfrac{1}{2}\beta_r \sum_{C_{r+1}} \Big(\prod_{C_r \subset \partial C_{r+1}} U(C_r) + \mathrm{cc}\Big) + \tfrac{1}{2}\lambda_{r-1} \sum_{C_r} \Big(U^q(C_r) \prod_{C_{r-1} \subset \partial C_r} \phi(C_{r-1}) + \mathrm{cc}\Big). \tag{14'}$$

The local gauge invariance of this action is

$$A \to A' = A + \mathrm{d}\theta, \qquad B \to B' = B - \theta q.$$

In the naive continuum limit, this action becomes equation (7) with the identification $g^2 \propto \lambda/\beta$. Another form of the action, which is very useful, is the Villain one

$$\exp S = \sum_{\{r,m\}} \exp\Big(-\tfrac{1}{2}\beta_r^{\mathrm{V}} \sum_{C_{r+1}} (\mathrm{d}A(C_{r+1}) + 2\pi r(C_{r+1}))^2$$
$$ - \tfrac{1}{2}\lambda_{r-1}^{\mathrm{V}} \sum_{C_r} (\mathrm{d}B(C_r) + qA(C_r) + 2\pi m(C_r)^2\Big) \tag{16}$$

where n and m are respectively $r+1$ and r integer valued forms. This action should be equivalent to the previous one in the following limits:

$$\beta_r^{\mathrm{V}} = \beta_r, \qquad \beta_r \to \infty,$$
$$\beta_r^{\mathrm{V}} = (2\ln(2/\beta_r))^{-1}, \qquad \beta_r \to 0,$$

and similarly for the λ-coupling.

We end this section with two remarks. The whole construction can be applied also to every abelian group as for example $\mathbb{Z}_N$: the case $N=2$ was already discussed in the pioneer work by Wegner (1971). Furthermore, the extension to non-abelian groups seems problematic, due to the lack of an ordering criterion for r-cells when $r>1$. In the abelian case, one can introduce Higgs fields whose radial component is not fixed (to 1 as in the present case). However, it is not clear what should be the corresponding continuum theory.

3. Correlation inequalities

For arbitrary r and $\lambda_{r-1}=0$ the phase structure is quite well understood, due to the work by Fröhlich and Spencer (1982a) (see also Orland 1982). Let us define the Wilson–Wegner operator as

$$W(\partial V_{r+1}) = \prod_{C_r \subset \partial V_{r+1}} U(C_r) \tag{17}$$

where V_{r+1} is an $(r+1)$-dimensional volume. The results of the previous reference can be summarised as follows: for (integer) dimensions, $d \leqslant r+2$ and $r \geqslant 1$ there is no phase transition (only a massive phase) and

$$\langle W(\partial V_{r+1})\rangle \leqslant \exp(-\alpha(\beta_r)|V_{r+1}|) \tag{18}$$

where $|V|$ is the volume of V and $\alpha>0$, i.e. external sources are 'confined' for every value of β_r. When $d \geqslant r+3$ there is a massive phase of small β_r where equation (18) holds and a massless phase of large β_r where instead of (18) we have

$$\langle W(\partial V_{r+1})\rangle \leqslant \exp(-\alpha'(\beta_r)|\partial V_{r+1}|). \tag{19}$$

For $r=0$ we have the Kosterlitz–Thouless transition already present in $d=2$ (Fröhlich and Spencer (1982b). Let us discuss the intuitive picture which emerges from these results. The thermal average of the Wilson–Wegner operator corresponds to evaluating the increase of the free energy in the system when an external conserved current (of rank r) is coupled to the gauge fields (of rank r). If equation (18) holds, then the free energy difference is proportional to the volume, which means that the external sources tend to collapse.

Correlation inequalities for these models can be derived following Ginibre's work (Ginibre 1970) with some slight modifications for the Villain form of the action. The inequalities we used are of the form

$$\langle fg\rangle - \langle f\rangle\langle g\rangle \geqslant 0 \tag{20}$$

with f and g belonging to the multiplicative cone generated by

$$\{\cos[(m, A)+(n, B)]\} \tag{21}$$

where m and n are integer valued r- and $(r-1)$-forms respectively. It is straightforward to deduce that the average of the Wilson–Wegner loop (17) is a monotone non-decreasing function of the couplings β and λ (see for some examples de Angelis *et al* 1977a, b and Guth 1980). Therefore it is immediate that $\langle W(\partial V_{r+1})\rangle$ is a monotone non-decreasing function of the dimensionality and in the case $\lambda_{r-1}=0$ this implies that the critical coupling $\bar{\beta}_r^{(d)}$, which is the border point between different behaviours (18) and (19), satisfies the inequality

$$\bar{\beta}_r^{(d+1)} \leqslant \bar{\beta}_r^{(d)}, \tag{22}$$

i.e. the Coulomb phase for the pure gauge theory is non-decreasing with the dimension.

Now we would like to have an inequality which relates critical couplings between models with different r's. For this reason, generalising a well known method (Brydges *et al* 1979a, b), we add to the original action an extra term

$$h_{\mathrm{G}} \sum_{\substack{C_r \subset \Lambda \\ \hat{o} \notin C_r}} \cos A(C_r) + h_{\mathrm{n}} \sum_{\substack{C_{r-1} \subset \Lambda \\ \hat{o} \notin C_{r-1}}} \cos B(C_{r-1}) \tag{23}$$

if the action has the Wilson form while

$$-h_{\mathrm{G}} \sum_{\substack{C_r \subset \Lambda \\ \hat{o} \notin C_r}} A^2(C_r) - h_{\mathrm{M}} \sum_{\substack{C_{r-1} \subset \Lambda \\ \hat{o} \notin C_{r-1}}} B^2(C_{r-1}) - \tilde{h}_{\mathrm{G}} \sum_{\substack{C_r \subset \Lambda \\ \hat{o} \in C_r}} A^2(C_r) - \tilde{h}_{\mathrm{M}} \sum_{\substack{C_{r-1} \subset \Lambda \\ \hat{o} \in C_{r-1}}} B^2(C_{r-1}) \tag{24}$$

when the action is the Villain one. In (23) and (24), $\hat{o} \notin C_r(C_{r-1})$ means that the direction $\hat{o}$ does not belong to the $C_r(C_{r-1})$ cell. For the Villain action the last two terms in (24) have been introduced so that the integration range of the A and B fields can be extended over the real line. Following a straightforward generalisation of Ginibre's work (Guth 1980, Elitzur *et al* 1979), it is easy to prove that $\langle W(\partial V_{r+1})\rangle_{h_{\mathrm{M}},h_{\mathrm{G}}}$ is a monotone non-decreasing function of h_{M} and h_{G} so that

$$\langle W(\partial V_{r+1})\rangle_{h_{\mathrm{M}}=h_{\mathrm{G}}=0} \leqslant \langle W(\partial V_{r+1})\rangle_{h_{\mathrm{M}}=h_{\mathrm{G}}=\infty} \tag{25}$$

(in general this inequality holds for any function belonging to the cone generated by the set in (21)).

In the RHS of (25) due to the $h \to \infty$ limit, we have

$$A(C_r)=0; \qquad B(C_{r-1})=0 \qquad \text{if } \hat{o} \notin C_r, C_{r-1},$$

which means that in such a limit the model becomes a product of $(d-1)$-dimensional models, one for each hyperplane with x_0 fixed, of the same type as the starting model but with r substituted by $r-1$ (for the Villain action we must put back $\tilde{h}=0$).

Now, if we consider a 'volume' $V_{r+1}=V_r \times T$ where T is the side along the $\hat{o}$ direction, it follows from (25) and from monotonic behaviour in β and λ that

$$\langle W(\partial V_{r+1})\rangle_d(\beta_r, \lambda_{r-1}) \leqslant \langle W(\partial V_r)\rangle_{d-1}^T(\beta'_{r-1}, \lambda'_{r-2}), \qquad \beta_r \in \beta'_{r-1}, \lambda_{r-1} \leqslant \lambda'_{r-2} \tag{26}$$

Putting $\lambda_{r-1}=\lambda'_{r-1}=0$, the same argument which led to (22) now gives

$$\bar{\beta}_{r-1}^{(d-1)} \leqslant \bar{\beta}_r^{(d)}. \tag{27}$$

This result combined with (22) also gives

$$\bar{\beta}_{r-1}^{(d)} \leqslant \bar{\beta}_r^{(d)}. \tag{28}$$

Equations (27) and (28) mean that on the pure gauge axis the critical coupling is non-decreasing in d for $d-r$ fixed and non-decreasing in r for d fixed.

Some simple consequences can be drawn from equation (26). In fact if the Higgs fields, B, have multiple charge, i.e. $q \neq 1$ in equations (14)–(16), then iterating (26) until the dimension becomes 2 and using a result due to Mack (1979), we obtain that the Wilson–Wegner loop decays exponentially with the volume $|V_{r+1}|$ for $d = r+1$†. Furthermore, because $\lambda = 0$ implies $\bar{\beta}_{r=1}^{(d=3)} = \infty$ for the Villain model (Göpfert and Mack 1982), we get from (27) that in $d = r+2$ and $\lambda = 0$, $\langle W(\partial V_{r+1})\rangle$ follows a volume law for all values of the coupling β_r† for the Villain action.

Another useful inequality follows from (26) when the limit $\lambda'_{r-2} \to \infty$ is considered. In this case the limit model of the RHS of (26) is the pure $\mathbb{Z}_q$ gauge model. It is easy to obtain, for the considered class of models, the diamagnetic inequality (Brydges *et al* 1979)

$$Z(A) \leqslant Z(A=0) \tag{29}$$

where $Z(A)$ is the partition function of the model where only the action of the matter field B interacting with an external source A is considered. Equation (29) holds in general when the Fourier coefficients of the cell action are non-negative. In fact, let

$$S(A,B) = \sum_{C_r} S_{C_r}[\mathrm{d}B(C_r) + qA(C_r)]$$

be the action and

$$\prod_r \mathrm{e}^{S(x_r)} = \sum_{n(C_r)\in\mathbb{Z}} \exp\Big(\mathrm{i}(n,x) + \sum_{C_r} f[n(C_r)]\Big)$$

with f a real function and n an integer valued r-form. Then

$$\begin{aligned} Z(A) = \int \mathscr{D}B\, \mathrm{e}^{S(A,B)} &= \sum_{n:\delta n=0} \exp\Big(\sum_{C_r} f[n(C_r)] + \mathrm{i}q(n,A)\Big) \\ &\leqslant \sum_{n:\delta n=0} \exp\Big(\sum_{C_r} f[n(C_r)]\Big) = Z(A=0). \end{aligned} \tag{30}$$

Thus the result follows because the actions (14) and (16) have non-negative Fourier coefficients, as is easy to prove.

It is worthwhile to mention also another form for the action of the pure gauge theory, i.e. the gaussian one

$$S(A) = -\tfrac{1}{2}\beta_r \sum_{C_{r+1}} (\mathrm{d}A(C_{r+1}))^2 = -\beta_r(\mathrm{d}A, \mathrm{d}A) \tag{31}$$

where now A is a real valued r-form. To avoid divergences due to the gauge invariance, we always think of the action (31) with a mass parameter which may be removed later. The matter action has either the Wilson form or Villain form as usual. Now the correlation inequalities (20) hold with f and g belonging to the multiplicative cone generated by the set (21) where, this time, m are real valued r-forms. (Brydges *et al* 1979). Using standard arguments, it follows that $\langle W(\partial V_{r+1})\rangle$ is a monotone non-decreasing function of β, λ and d. Furthermore, when $\lambda = 0$ the inequality (22) still

† Strictly speaking, we obtain a decay faster than the volume. However, it is not difficult to generalise a result by Seiler (1978) and Simon and Yaffe (1982) which states that $\langle W(\partial V_{r+1})\rangle$ is bounded from below by a volume law so that the decay follows a volume law.

holds and, adding to the action a term of the type

$$-h_G \sum_{\substack{C_r \subset \Lambda \\ \hat{0} \notin C_r}} A^2(C_r)$$

and an analogous one for the matter field B (see (23) and (24)), we get also the inequality (26) in the same way as before. Now the case $\lambda = 0$ is trivial, because the action is quadratic in the field A. Thus, in this case we consider (26) for $\lambda_{r-1} = \lambda'_{r-2} \to \infty$. In such a limit the field A becomes

$$A \xrightarrow[\lambda \to \infty]{} (2\pi/q)n$$

where n is an integer valued r-form while the action becomes

$$S(n) = -\beta_r(2\pi/q)^2(\mathrm{d}n, \mathrm{d}n) \tag{32}$$

which is similar to the action of a massless free field theory, but now the field takes only integer values. Therefore, we get the inequalities (27) and (28) for a pure gauge theory with integer valued r-forms.

The whole set of inequalities that we have obtained by generalising arguments which were known for the gauge theory could be useful for understanding the model. The $r = 1$ case was discussed by Fradkin and Shenker (1979) and it is important also for the case $r > 1$. For such a purpose, we stress that it is easy to generalise the Osterwalder–Seiler proof of the existence of the cluster expansion which converges uniformly in Λ when the action has the Wilson or Villain form. This implies mass gap and analyticity of the expectations of local observables in the region of the (β, λ) plane where the expansion converges (Osterwalder and Seiler 1978). Such a result can be used (Fradkin and Shenker 1979) to conclude the non-existence of the phase boundary between the Higgs regime (β, λ large) and the confinement regime (β, λ small) if the Higgs fields B have fundamental charge ($q = 1$).

4. θ-vacua

Now we would like to generalise θ-vacua to the gauge differential forms of rank r.

In the continuum, θ-vacua are related to the existence of a topological charge in the theory (see e.g. Seiler 1981). In the usual case of the one-form gauge fields, this happens in $d = 2$ for the U(1) group and in $d = 4$ for SU(N) groups where the topological charges are respectively the first Chern number $C_1 \equiv -(2\pi)^{-1} \int_{s^2} F$ and the second Chern number $C_2 = (8\pi)^{-1} \int_{S^4} \mathrm{Tr}\, F \wedge F$. For the U(1) valued r-form gauge field it is possible to define an analogue of the first Chern number in $d = r+1$ as $C_1^{(r)} = -(2\pi)^{-1} \int_{s^{r+1}} F$ where F is now an $(r+1)$-form (Marchetti 1982), because F, being an $(r+1)$-form on S^{r+1}, gives rise naturally to a two-form on $r-1$-iterated loop space, and, simultaneously, the sphere S^{r+1} can be considered as a two-dimensional closed surface in the $r-1$-time iterated loop space.

On the lattice Λ, $\int F$ corresponds to

$$\sum_{C_{r+1} \subset \Lambda} \mathrm{d}A(C_{r+1}). \tag{33}$$

In the continuum, θ-vacua for the U(1) group are obtained by adding the term $(\mathrm{i}\theta/2\pi)C_1$ to the action, which corresponds to adding the term

$(\mathrm{i}\theta/2\pi)/\sum_{C_{r+1}\subset\Lambda}\mathrm{d}A/(C_{r+1})$ to the lattice action which is gaussian in the A fields. The existence of θ-vacua can be related to the existence of an 'instanton' like configuration for the r-forms in $r+1$ dimensions, which are the well known Nielson–Olesen vortex (Nielsen and Olesen 1973) for $r=1$ and $d=2$. In particular, for $r-3$ and $d=4$ these configurations are related to the Yang–Mills instantons through the identification of $\partial_\sigma A_{\mu\nu\rho}\varepsilon^{\mu\nu\rho\sigma}$ with the topological charge density $\mathrm{Tr}\,\varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$, where $F^{\mu\nu}$ is now the Yang–Mills field strength (Lüscher 1978).

Before studying the properties of the model we verify that, in $d=r+1$, the gaussian and Villain forms of the action for the gauge field are equivalent if the matter field action is periodic (this point will be useful because we shall need some correlation inequalities of the gaussian model). In fact, working in the axial gauge, which means that $A_{\mu_0\mu_1\ldots\mu_{r-1}|\mu_0=\hat{o}}=0$ for r-forms, we have

$$\mathrm{d}A(C_{r+1})=A(C_r)-A(C_r-\hat{o}) \tag{34}$$

where $C_r, C_r-\hat{o}\in\partial C_{r+1}$. Then for any observable which is periodic in the A field, the integration range for the A field can be restricted to $(0,2\pi)$ and the exponentiated gauge action becomes

$$\mathrm{e}^S=\sum_{\{n\}}\exp\Big(-\tfrac{1}{2}\beta\sum_{C_{r+1}}(\mathrm{d}A(C_{r+1})+2\pi\,\mathrm{d}n(C_{r+1}))\Big)^2 \tag{35}$$

where n is an integer valued r-form in the axial gauge (34). Since in (35) $\mathrm{d}n$ is the difference of two integers without constraints, it can be replaced by $m\in\mathbb{Z}$ (determined by $\mathrm{d}n$ apart from a global constant integer) so that (35) becomes the Villain form of the gauge action.

Now, let us perform the duality transformation for the Villain model after fixing the gauge $B=0$ (unitary gauge):

$$\begin{aligned}
Z_{\theta,\Lambda}&=\int\mathscr{D}A\sum_{\{m\}}\sum_{\{n\}}\exp\Big[-\tfrac{1}{2}\beta\sum_{C_{r+1}\subset\Lambda}(\mathrm{d}A(C_{r+1})+2\pi m(C_{r+1}))^2\\
&\qquad-\tfrac{1}{2}\lambda\sum_{C_r\subset\Lambda}(A(C_r)+2\pi nC_r))^2+\frac{\mathrm{i}\theta}{2\pi}\sum_{C_{r+1}\subset\Lambda}(\mathrm{d}A(C_{r+1})+2\pi m(C_{r+1}))\Big]\\
&=\text{constant}\int\mathscr{D}A\sum_{\{m,n\}}\exp\Big[\mathrm{i}\Big(m,\mathrm{d}A-\frac{\mathrm{i}\theta_1}{2\pi\beta}\Big)-\frac{\theta^2}{8\pi^2\beta}-\frac{1}{2\beta}(m,m)\Big]\\
&\qquad\times\exp\Big(\mathrm{i}(n,A)-\frac{1}{2\lambda}(n,n)\Big)\\
&=\text{constant}\exp\Big(-\frac{\theta^2}{8\pi^2\beta}\Big)\sum_{\{m,n\};\delta m+n=0}\exp\Big(m\frac{\theta\mathbb{1}}{2\pi\beta}\Big)\\
&\qquad\times\exp\Big(-\frac{1}{2\beta}(m,m)-\frac{1}{2\lambda}(n,n)\Big)\\
&=\text{constant}\sum_{\{m\}}\exp\Big[-\frac{1}{2\lambda}(\delta m,\delta m)-\frac{1}{2\beta}\Big(m+\frac{\theta\mathbb{1}}{2\pi},m+\frac{\theta\mathbb{1}}{2\pi}\Big)\Big].
\end{aligned} \tag{36}$$

Thus, we have that $Z_{\theta,\Lambda}$ is periodic in θ. Now we would like to prove the existence of the thermodynamic limit for the energy density

$$\varepsilon(\theta)=\lim_{\Lambda\nearrow\mathbb{Z}^{r+1}}\frac{1}{|\Lambda|}\ln\frac{Z_{\theta\Lambda}}{Z_{0\Lambda}}=\lim_{\Lambda\nearrow\mathbb{Z}^{r+1}}\ln\left\langle\exp\left(\frac{\mathrm{i}\theta}{2\pi}\sum_{C_{r+1}\subset\Lambda}\mathrm{d}A(C_{r+1})\right)\right\rangle_\Lambda \tag{37}$$

where the subindex Λ in the last average means that the Boltzmann weight is restricted to Λ.

First, we notice that the θ-states have the Osterwalder–Schrader positivity (reflection positivity can be proved, for the model considered in this paper, generalising standard arguments (Osterwalder and Seiler 1978, Seiler 1981) in a straightforward way). Thus, we apply the Schwarz inequality to the Osterwalder–Schrader inner product to obtain the following inequality:

$$\left\langle\exp\left(\frac{\mathrm{i}\theta}{2\pi}\sum_{C_{r+1}\subset\Lambda'}\mathrm{d}A(C_{r+1})\right)\right\rangle_\Lambda\leqslant\left[\left\langle\exp\left(\frac{\mathrm{i}\theta}{2\pi}\sum_{C_{r+1}\subset\Lambda}\mathrm{d}A(C_{r+1})\right)\right\rangle_\Lambda\right]^{1/2} \tag{38}$$

where Λ' is one half of the lattice Λ. Now, using inequality (20) for the multiplicative cone generated by the set (21), with m a real valued r-form, we have that the LHS of (38) is not decreasing with Λ, and thus it can be replaced by the average taken on the lattice Λ'. Iterating (38), we finally obtain

$$\frac{1}{|\Lambda'|}\ln\left\langle\exp\left(\frac{i\theta}{2\pi}\sum_{C_{r+1}\subset\Lambda'}\mathrm{d}A(C_{r+1})\right)\right\rangle_{\Lambda'}\leqslant\frac{1}{|\Lambda|}\ln\left\langle\exp\left(\frac{i\theta}{2\pi}\sum_{C_{r+1}\subset\Lambda}\mathrm{d}A(C_{r+1})\right)\right\rangle_\Lambda\leqslant 0 \tag{39}$$

where $|\Lambda|=2^n|\Lambda'|$ and $n\in\mathbb{N}$. Then, it is not difficult to conclude the existence of the limit (37). Furthermore, from (36) and due to the monotonic behaviour in the coupling in the last average of (37), taking $\lambda\to+\infty$ we get

$$\varepsilon(\theta)\leqslant\ln\frac{\sum_{-\infty}^{+\infty}{}_m\exp[-(1/2\beta)(m+\theta/2\pi)^2]}{\sum_{-\infty}^{+\infty}{}_m\exp(-m^2/2\beta)}<0,\qquad\theta\neq 0\bmod 2\pi.$$

Along the same lines, it is very easy to prove

$$|\langle W_{\theta'}(\partial V_{r+1})\rangle_\theta|\leqslant\exp[(\varepsilon(\theta+\theta')-\varepsilon(\theta))|V_{r+1})]$$

where

$$(W_{\theta'}(\partial V_{r+1})=\prod_{C_r\in\partial V_{r+1}}\exp[\mathrm{i}\theta' A(C_r)/2\pi]$$

and the subscript θ means that the θ term is present in the action.

5. Conclusion

In this paper we have obtained generalised inequalities for a gauge field of rank r in a d-dimensional lattice which can be useful for drawing tentative phase diagrams for models of interacting r- and $(r-1)$-forms.

The existence of a region of analyticity in the coupling constant space for local observables suggests phase diagrams which are similar to the ones considered by Fradkin and Schenker (1979). As proved by Fröhlich and Spencer, the pure gauge theory has a lower critical integer dimensionality d_c^L equal to $r+3(r>1)$. Regarding the $\mathbb{Z}_q$ model, using duality arguments and the above-mentioned inequalities, it is easy to see that $\mathrm{d}_c^L=r+2$.

The analysis of θ-vacua in $d = r+1$ on the lattice confirms the classification given by Isham (1981) in the continuum (Marchetti and Percacci 1982) and gives rise to a natural extension of properties of the scalar $(\text{QED})_2$.

Note added in proof. With regard to the inequality (26), it was proved by Maritan and Stella (1983) that there exists a particular lattice where it is satisfied as an equality.

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